# The weak amalgamation property

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# The setup

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$$\sigma\mathfrak{K} = \Big\{ \bigcup_{n\in\omega} X_n \colon \{X_n\}_{n\in\omega} \text{ is a chain in } \mathfrak{K} \Big\}.$$

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Next, Adam chooses $A_{2k+1} \in \mathfrak{K}$ together with an embedding $A_{2k}^{2k+1} \colon A_{2k} \to A_{2k+1}$. And so on...
The result of a play is a chain $\vec{a}$:

$$A_0 \xrightarrow{a_0^1} A_1 \longrightarrow \cdots \longrightarrow A_{2k-1} \xrightarrow{a_{2k-1}^{2k}} A_{2k} \longrightarrow \cdots$$

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We say that $U \in \sigma\mathfrak{K}$ is $\mathfrak{K}$-generic if Adam has a strategy in the Banach-Mazur game BM $(\mathfrak{K})$ such that the union of the resulting chain $\vec{a}$ is always isomorphic to $U$, no matter how Eve plays.

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**Proof.**

The rules for Eve and Adam are the same. □

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We say that $\mathfrak{K}$ has amalgamations at $Z \in \mathfrak{K}$ if for every embeddings $f \colon Z \to X$, $g \colon Z \to Y$ there exist $w \in \mathfrak{K}$ and embeddings $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f = g' \circ g$.

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We say that $\mathfrak{K}$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \mathfrak{K}$.

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The class $\mathfrak{K}$ is directed if for every $X, Y \in \mathfrak{K}$ there is $V \in \mathfrak{K}$ such that both $X$ and $Y$ embed into $V$.

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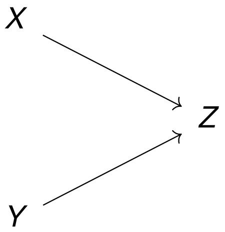

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We say that $\mathfrak{K}$ has the cofinal amalgamation property (CAP) if for every $Z \in \mathfrak{K}$ there is an embedding $e \colon Z \to Z'$ such that $\mathfrak{K}$ has amalgamations at $Z'$.

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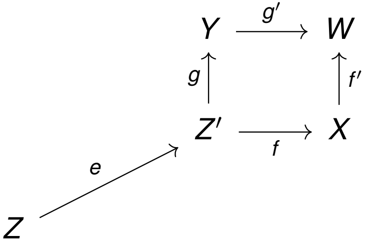
## Definition (Ivanov, 1999)

We say that $\mathfrak{K}$ has the weak amalgamation property (WAP) if for every $Z \in \mathfrak{K}$ there is an embedding $e \colon Z \to Z'$ with $Z' \in \mathfrak{K}$, such that for every embeddings $f \colon Z' \to X$, $g \colon Z' \to Y$ there exist embeddings $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f \circ e = g' \circ g \circ e$.

# CAP and WAP



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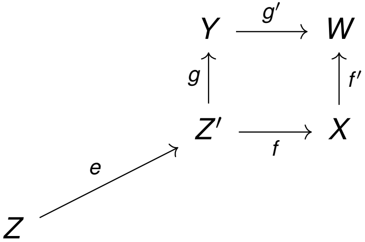


## Proposition

*Finite graphs of vertex degree $\leqslant 2$ have the CAP.*

# Main results

**Theorem (Krawczyk & K. 2016)**

*Let $\mathfrak{K}$ be a countable directed category of finitely generated models. The following conditions are equivalent:*

(a) *There exists a $\mathfrak{K}$-generic model.*

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**Theorem (Krawczyk & K. 2016)**

*Let $\mathfrak{K}$ be as above and let $U \in \sigma\mathfrak{K}$. The following properties are equivalent:*

(a) *$U$ is $\mathfrak{K}$-generic.*

(b) *Eve does not have a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$.*

# Examples...

A. Krawczyk, W. Kubiś, *Games on finitely generated structures*, preprint, arXiv:1701.05756

A. Krawczyk, A. Kruckman, W. Kubiś, A. Panagiotopoulos, *Examples of weak amalgamation classes*, preprint

W. Kubiś, *Weak Fraïssé categories*, preprint, arXiv:1712.03300